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A Note on Einstein-Sasaki Metrics in $D \geq 7$

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ABSTRACT

In this paper, we obtain new non-singular Einstein-Sasaki spaces in dimensions $D \geq 7$. The local construction involves taking a circle bundle over a $(D-1)$-dimensional Einstein-Kähler metric that is itself constructed as a complex line bundle over a product of Einstein-Kähler spaces. In general the resulting Einstein-Sasaki spaces are singular, but if parameters in the local solutions satisfy appropriate rationality conditions, the metrics extend smoothly onto complete and non-singular compact manifolds.

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1 Introduction

Einstein metrics admitting Killing spinors are of considerable interest in string theory and M theory, since they can provide supersymmetric backgrounds of relevance to the AdS/CFT correspondence [1]. For the case of an Einstein-Sasaki space X_{2n+3}, a solution with the geometry $\mathrm{AdS}_d \times X_{2n+3}$ is expected to be dual to a $d-1$ dimensional superconformal field theory with reduced supersymmetry. Such solutions arise in the near-horizon limit of certain p-branes located at the tip of the corresponding Calabi-Yau cone $C(X_{2n+3})$ [2, 3, 4, 5, 6]. For example, an M2-brane on a cone with special holonomy $SU(4)$ interpolates between $\mathrm{AdS}_4 \times X_7$ and $\mathrm{Minkowski}_3 \times C(X_7)$, which implies that there is an RG-flow in the quantum field theoretical picture [5].

Until recently, the known explicit Einstein-Sasaki metrics were relatively sparse. Well-known examples are the round sphere in any odd dimension, and the sphere with the non-standard "squashed" Einstein metric in dimensions $D = 4n - 1$, described as a coset $Sp(n+1)/Sp(n)$. Other examples include the five-dimensional $T^{1,1}$ space (which is topologically $S^2 \times S^3$), and higher-dimensional analogues. Aside from these isolated examples, which are all homogeneous, there were various existence proofs for further inhomogeneous Einstein-Sasaki metrics, including, for example, 13 on $S^2 \times S^3$ [7]. The collection of examples increased dramatically recently, with the explicit construction of infinitely many inhomogeneous non-singular Einstein-Sasaki metrics in all odd dimensions $D = 2n + 3 \geq 5$ [8, 9].

An Einstein-Sasaki metric can always be viewed as a circle bundle over an Einstein-Kähler base space, written as

$$d\hat{s}^2 = (d\psi' + 2\mathcal{A}_{(1)})^2 + ds^2 \,, \tag{1}$$

where $d\mathcal{A}_{(1)}$ is proportional to the Kähler form for ds^2. (See, for example, [10] for an explicit discussion of this.) The Einstein-Kähler bases ds^2 used in [8, 9] are the class of such metrics that were constructed in [13, 14]. These Einstein-Kähler metrics were themselves obtained as two-dimensional bundles over Einstein-Kähler base metrics $d\tilde{s}^2$ of dimension $2n$:

$$ds^2 = \frac{d\rho^2}{U(\rho)} + \rho^2 U(\rho)\,(d\tau' + \mathcal{B}_{(1)})^2 + \rho^2\, d\tilde{s}^2 \,, \tag{2}$$

where $d\mathcal{B}_{(1)}$ is proportional to the Kähler form for $d\tilde{s}^2$.

The $2n+2$-dimensional Einstein-Kähler metrics obtained in [13, 14] are generally singular. However, this need not necessarily imply that the Einstein-Sasaki metrics on circle

bundles over them are singular. Indeed, the main subtlety in the construction of [8, 9] consists in showing that the Einstein-Sasaki metrics can, for suitable choices of parameters, be extended smoothly onto compact manifolds, even though the Einstein-Kähler base spaces by themselves are singular.[1]

In this paper, we obtain further examples of non-singular Einstein-Sasaki metrics, by generalising the construction described above. Specifically, we do this by extending the construction of $2n+2$-dimensional Einstein-Kähler metrics to cases where the $2n$-dimensional base metric is a product of Einstein-Kähler factors $d\tilde{s}_i^2$, rather than a single one;

$$ds^2 = dt^2 + c^2\,(d\tau' + \mathcal{B}_{(1)})^2 + \sum_i a_i^2\, d\tilde{s}_i^2\,, \tag{3}$$

where c and a_i are function of the radial variable t. Although we find that the metrics ds^2 are generally singular, in certain cases the Einstein-Sasaki metric $d\hat{s}^2$ given by (1) can extend smoothly onto a non-singular manifold, even though the Einstein-Kähler base space is singular.

This paper is organized as follows. In section 2, we give a detailed exposition of the construction for the case where ds^2 is a six-dimensional Einstein-Kähler metric constructed as a two-dimensional bundle over a product $S^2 \times S^2$ base. We obtain seven-dimensional Einstein-Sasaki metrics, which is the lowest dimensionality for which the generalisation extends beyond the results in [9]. In section 3, we generalize the construction to higher dimensions by using a base space composed of a product of an arbitrary number of Einstein-Kähler spaces, each of arbitrary even dimensionality. Conclusions are presented in section 4.

2 Seven-Dimensional Einstein-Sasaki Metrics

2.1 The six-dimensional Einstein-Kähler base

We begin by constructing six-dimensional Einstein-Kähler metrics of the form

$$ds_6^2 = dt^2 + c^2\,(d\tau' + \mathcal{B}_{(1)})^2 + a^2 d\Omega_2^2 + b^2 d\tilde{\Omega}_2^2\,, \tag{4}$$

where

$$d\Omega_2^2 = d\theta^2 + \sin^2\theta d\phi^2\,, \qquad d\tilde{\Omega}_2^2 = d\tilde{\theta}^2 + \sin^2\tilde{\theta} d\tilde{\phi}^2\,, \tag{5}$$

[1] An analogous approach can be used to demonstrate that the G_2 holonomy metrics constructed in [11] as $SU(2)$ bundles over singular self-dual Einstein 4-spaces are also complete and non-singular [12].

and the connection $\mathcal{B}_{(1)}$ is such that

$$d\mathcal{B}_{(1)} = p\,\Omega_{(2)} + q\,\tilde\Omega_{(2)} \equiv -p\,dA_{(1)} - q\,d\tilde A_{(2)}\,, \tag{6}$$

where $\Omega_{(2)}$ and $\tilde\Omega_{(2)}$ are the volume forms of the two unit 2-spheres. We shall take

$$A_{(1)} = \cos\theta\,d\phi\,, \qquad \tilde A_{(1)} = \cos\tilde\theta\,d\tilde\phi\,. \tag{7}$$

In order that the circle bundle over $S^2\times S^2$ be well defined, the ratio p/q must be rational, so that the periods dictated for τ' by the consideration of the bundle over each S^2 factor are commensurate. By a rescaling of τ, one can then, without loss of generality, choose p and q to be relatively-prime integers. They characterise the winding numbers of the circle bundle over the two 2-spheres of the base. Without loss of generality, p and q can be taken to be positive. In fact, when we construct the Einstein-Sasaki 7-metrics as circle bundles over these 6-metrics, it will turn out that the ratio p/q no longer needs to be rational.

To impose the Kähler condition on (4), we begin by choosing a complex structure for which the Kähler 2-form is

$$J = c\,dt\wedge(d\tau' + \mathcal{B}_{(1)}) + a^2\,\Omega_{(2)} + b^2\,\tilde\Omega_{(2)}\,. \tag{8}$$

A necessary condition for Kählerity is then that $dJ=0$, implying

$$\frac{\dot a}{a} = \frac{p\,c}{2a^2}\,, \qquad \frac{\dot b}{b} = \frac{q\,c}{2b^2}\,. \tag{9}$$

In fact, in this case it is easily established that this already implies that J is covariantly constant, $\nabla_k J_{ij}=0$, and thus (9) constitutes necessary and sufficient conditions for (4) to be Kähler.

Now, we impose the additional requirement that (4) be an Einstein metric. We shall choose the normalisation $R_{\mu\nu} = 5g^2\,g_{\mu\nu}$. Calculating the Ricci tensor for (4), we find that the Einstein condition implies

$$\begin{aligned}
-\frac{\ddot a}{a} - \frac{\dot a^2}{a^2} - \frac{2\dot a\,\dot b}{a\,b} - \frac{\dot a\,\dot c}{a\,c} - \frac{p^2\,c^2}{2a^4} + \frac{1}{a^2} &= 5g^2\,,\\
-\frac{\ddot b}{b} - \frac{\dot b^2}{b^2} - \frac{2\dot a\,\dot b}{a\,b} - \frac{\dot b\,\dot c}{b\,c} - \frac{q^2\,c^2}{2b^4} + \frac{1}{b^2} &= 5g^2\,,\\
-\frac{\ddot c}{c} - \frac{2\dot a\,\dot c}{a\,c} - \frac{2\dot b\,\dot c}{b\,c} + \frac{p^2\,c^2}{2a^4} + \frac{q^2\,c^2}{2b^4} &= 5g^2\,,\\
-\frac{2\ddot a}{a} - \frac{2\ddot b}{b} - \frac{\ddot c}{c} &= 5g^2\,.
\end{aligned} \tag{10}$$

Substituting (9) into (10), we find that

$$\frac{\dot c}{c} = -\frac{p\,c}{2a^2} - \frac{q\,c}{2b^2} + \frac{1-5g^2\,a^2}{p\,c}\,, \tag{11}$$

together with an algebraic constraint

$$p - q + 5g^2(q\,a^2 - p\,b^2) = 0\,. \tag{12}$$

Summarising our results so far, we have shown that (4) is an Einstein-Kähler metric if the following system of equations is satisfied:

$$\begin{aligned}
&\frac{\dot a}{a} = \frac{p\,c}{2a^2}\,, \qquad \frac{\dot b}{b} = \frac{q\,c}{2b^2}\,, \qquad \frac{\dot c}{c} = -\frac{p\,c}{2a^2} - \frac{q\,c}{2b^2} + \frac{1-5g^2\,a^2}{p\,c}\,,\\
&p - q + 5g^2(q\,a^2 - p\,b^2) = 0\,.
\end{aligned} \tag{13}$$

To solve (13), we introduce a new radial variable r such that $dr = c\,dt$, leading straightforwardly to

$$\begin{aligned}
a^2 &= p\,r + \ell_1\,, \qquad b^2 = q\,r + \ell_2\,,\\
c^2 &= \frac{2}{a^2\,b^2}\int_0^r a^2\,b^2\,\Big(\frac{1-5g^2a^2}{p}\Big)\,dr'\\
&= -\frac{r}{12p\,(p\,r+\ell_1)(q\,r+\ell_2)}\,\Big[15g^2\,p^2\,q\,r^3 + 4p\,[5g^2\,(p\,\ell_2 + 2q\,\ell_1 - q)]\,r^2\\
&\quad + 6[5g^2\ell_1\,(q\,\ell_1 + 2p\,\ell_2) - (q\,\ell_1 + p\,\ell_2)]\,r + 60g^2\,\ell_1^2\,\ell_2 - 12\ell_1\,\ell_2\Big]\,.
\end{aligned} \tag{14}$$

The algebraic constraint in (13) becomes

$$p - q + 5g^2\,(q\ell_1 - p\ell_2) = 0\,. \tag{15}$$

Note that the choice of origin for r is arbitrary, since r does not appear explicitly in the equations. We have exploited this by choosing the lower limit of the integration in (14) to be at $r = 0$. This implies that the two integration constants ℓ_1 and ℓ_2 in (14) are non-trivial parameters.

2.2 The seven-dimensional Einstein-Sasaki metrics

Having obtained the six-dimensional Einstein-Kähler base metrics ds^2, we can now proceed to the construction, via (1), of the seven-dimensional Einstein-Sasaki metrics. With the Kähler form J for ds^2 given by (8), we can introduce the following potential $\mathcal{A}_{(1)}$, such that $J = d\mathcal{A}_{(1)}$:

$$\mathcal{A}_{(1)} = r\,d\tau' - a^2\,A_{(1)} - b^2\,\tilde A_{(1)}\,. \tag{16}$$

The Einstein-Sasaki 7-metric is then given by

$$d\hat s_7^2 = k^2\,(d\psi' + 2\mathcal{A}_{(1)})^2 + ds_6^2\,, \tag{17}$$

where the Einstein condition implies that we must have

$$k^2 = \tfrac{5}{8}\, g^2 \,. \tag{18}$$

The Ricci tensor of $d\hat{s}^2$ then satisfies $\hat{R}_{ab} = \frac{15}{4} g^2\, \hat{g}_{ab}$. It is convenient to choose a normalisation such that $\hat{R}_{ab} = 6\hat{g}_{ab}$, implying that $g^2 = 8/5$. Hence, $k = 1$ and the six-dimensional Einstein-Kähler metric satisfies $R_{ij} = 8g_{ij}$.

The six-dimensional Einstein-Kähler metrics ds_6^2 that we obtained in section 2.1 generally do not extend smoothly onto complete non-singular manifolds. We take the radial coordinate r to range between two zeros of the metric function $c(r)$, $r_- \le r \le r_+$, for which $a(r)$ and $b(r)$ remain non-vanishing. In order for the metric to have a smooth extension, $c(r)$ must approach zero at the two endpoints at the appropriate rate. This rate determines the period required for ψ in order that the metric in the (r, ψ) plane extend smoothly onto $\mathbb{R}^2$ at the "origin" $r = r_-$ or $r = r_+$. In order for the metric to extend globally onto a smooth manifold, the periods for ψ at the two endpoints need to be identical, and must be consistent with that allowed by the requirement of well-definedness of the 1-form $(d\tau' + \mathcal{B}_{(1)})$. These multiple criteria are, in fact, not fulfilled for the six-dimensional Einstein-Kähler metrics ds_6^2.

Nevertheless, as we mentioned earlier, this does not necessarily imply that the seven-dimensional Einstein-Sasaki metric $d\hat{s}_7^2$ on the circle bundle over ds_6^2 is singular. We therefore need to study the global structure of $d\hat{s}_7^2$ carefully, using techniques of the kind described in [8, 9]. We find that it is appropriate to define new fibre coordinates τ and ψ, related to τ' and ψ' by

$$\psi' = 2\tau \,, \qquad \tau' = \frac{8\tau - \psi}{8\beta} \,. \tag{19}$$

In terms of these, we can re-express the Einstein-Sasaki metric (17) as

$$\begin{aligned} d\hat{s}_7^2 \;&=\; \frac{dr^2}{c^2} + \frac{c^2}{16(c^2 + 4(\beta + r)^2)} (d\psi - A_{(1)} - \widetilde{A}_{(1)})^2 + a^2\, d\Omega_2^2 + b^2\, d\widetilde{\Omega}_2^2 \\ &\quad + \frac{c^2 + 4(\beta + r)^2}{\beta^2} \Big(d\tau - \ell_1 A_{(1)} - \ell_2 \widetilde{A}_{(1)} - \frac{c^2 + 4r(\beta + r)}{8(c^2 + 4(\beta + r)^2)} (d\psi - A_{(1)} - \widetilde{A}_{(1)}) \Big)^2 \,, \end{aligned} \tag{20}$$

where

$$\beta = \frac{8\ell_1 - 1}{8p} = \frac{8\ell_2 - 1}{8q} \,. \tag{21}$$

(This equation, which follows from a global consideration, will be discussed below.) Note that (21) is consistent with the algebraic constraint (15), since we have made the normalisation choice $g^2 = 8/5$.

The metric has a rescaling symmetry under which $p \to \lambda\, p$, $q \to \lambda q$ and $r \to r/\lambda$. Thus only the ratio p/q of the parameters p and q is non-trivial. This ratio is determined in terms of ℓ_1 and ℓ_2 by the algebraic constraint (15), which is rewritten, after setting $g^2 = 8/5$, in (21). It should be emphasised that in the discussion of the six-dimensional Einstein-Kähler space in section 2.1, p/q had to be rational in order that the circle bundle over $S^2 \times S^2$ was non-singular, but that requirement is, as we shall see, no longer necessary when considering the regularity of the Einstein-Sasaki space. The parameters p and q need only satisfy the constraint (21). Thus as far as local considerations are concerned, we have a family of Einstein-Sasaki metrics described by the two non-trivial real parameters ℓ_1 and ℓ_2. They characterise the size of the S^2 bolts.

In our solution (14) for the metric functions a, b and c, we chose an integration constant so that $r = 0$ is one of the zeros of the function $c(r)$. Thus we take r to lie in the range $0 \le r \le r_+$, where r_+ is the smallest positive zero of $c(r)$. (We can always, without loss of generality, choose to consider r non-negative.) The functions $a(r)$ and $b(r)$ should remain non-zero in the entire range $0 \le r \le r_+$. We therefore have the following conditions:

$$
\begin{aligned}
&\ell_1 > 0\,, \qquad \ell_2 > 0\,, \\
&r_+ > 0\,, \qquad c(r_+) = 0\,, \qquad (c^2)'(0) > 0\,.
\end{aligned} \tag{22}
$$

Note that from the last condition, it follows that $c^2(r) > 0$ for $0 < r < r_+$, and that $(c^2)'(r_+) < 0$.

To study the global structure, we first consider the base manifold, whose metric is given by the terms appearing in the first line of (20) (i.e. the terms orthogonal to the τ fibres). It can be viewed as an S^2 bundle over $S^2 \times S^2$, where the S^2 bundle is coordinatised by r and ψ. Without loss of generality, we may assume that $0 \le p \le q$, in which case the positivity of a^2 and b^2, together with the conditions (22), implies that

$$
\tfrac18 (1 - \frac{p}{q}) < \ell_1 < \tfrac18\,, \qquad 0 < \ell_2 < \tfrac18\,, \tag{23}
$$

The Killing vector $\partial/\partial\psi$ degenerates at the points $r = 0$ and $r = r_+$ where $c(r)$ vanishes. In order for the metric to extend smoothly onto these points, it is necessary that the period of ψ be commensurate with the slope of $c^2(r)$. Specifically, if $c^2(r)$ has slope $(c^2)'(r_0) = K(r_0)$ at one of these endpoints, say $r = r_0$, then writing $c^2(r) \sim K(r_0)\,(r - r_0)$ nearby, and defining $\rho^2 = r - r_0$, we see from (20) that in the (r, ψ) frame we shall have

$$
ds^2 \sim \frac{4}{K} \Big(d\rho^2 + \frac{K^2\, \rho^2}{256(\beta + r_0)^2}\, d\psi^2 \Big)\,. \tag{24}
$$

This implies that ψ must have period

$$\Delta\psi = \left|\frac{32\pi(\beta+r_0)}{K(r_0)}\right| . \tag{25}$$

The periods determined by these conditions at $r_0=0$ and $r_0=r_+$ will agree if (see (14))

$$\beta\Big(r_+ + \frac{8\ell_1-1}{8p}\Big) = (\beta+r_+)\Big(\frac{8\ell_1-1}{8p}\Big) , \tag{26}$$

which is satisfied if

$$\beta = \frac{8\ell_1-1}{8p} . \tag{27}$$

This, together with the relation in terms of ℓ_2 implied by (15), gives the conditions appearing in (21). Note that (25) now implies that ψ has period 2π. [2]

The $U(1)$ fibre parameterised by the coordinate τ in (20) never collapses, and so it follows that the period of τ is governed only by the connection on the fibre, given by

$$d\tau - \ell_1 A_{(1)} - \ell_2 \widetilde{A}_{(1)} - \frac{c^2+4r(\beta+r)}{8(c^2+4(\beta+r)^2)}(d\psi - A_{(1)} - \widetilde{A}_{(1)}) . \tag{28}$$

The global structure can be examined by looking at all the cycles at $r=0$ and $r=r_+$ where c^2 vanishes. They are given by

$$\begin{array}{lllll}
r=0: & A_{(1)}: & 2\pi\,\ell_1 , & \widetilde{A}_{(1)}: & 2\pi\,\ell_2 , \\
r=r_+: & A_{(1)}: & 2\pi\,(\frac{r_+}{8(\beta+r_+)}-\ell_1) , & \widetilde{A}_{(1)}: & 2\pi\,(\frac{r_+}{8(\beta+r_+)}-\ell_2) , \\
 & d\psi: & 2\pi\,(\frac{r_+}{8(\beta+r_+)}) . & &
\end{array} \tag{29}$$

For the expression in (28) to be globally extendible, the ratios of the above quantities must all be rational. Thus there are two independent requirements, namely

$$\frac{\ell_1}{\ell_2} = \alpha \equiv \text{rational number} , \qquad \frac{r_+}{(\beta+r_+)\ell_1} = \gamma \equiv \text{rational number} . \tag{30}$$

One then solves the cubic polynomial for r_+ that follows from setting $c(r_+)^2=0$ in (14). Using the two rationality conditions (30), together with (15), enables us to express ℓ_2 purely in terms of α and γ:

$$\begin{aligned}
0 &= 1536\alpha^2\,\gamma^3\,\ell_2^4 + 64\alpha\,\gamma^2\Big(\alpha\,(\gamma-96) - 30\gamma\Big)\ell_2^3 \\
&+ 8\gamma\Big(32\alpha^2\,(36-\gamma) + 27\gamma^2 + 4\alpha\,\gamma\,(72+7\gamma)\Big)\ell_2^2 \\
&+ \Big(384\alpha^2\,(\gamma-16) - 32\alpha\,\gamma\,(24+7\gamma) - \gamma^2\,(192+29\gamma)\Big)\ell_2 \\
&+ 48\alpha\,(16-\gamma) + 16\gamma\,(2\gamma-3) .
\end{aligned} \tag{31}$$

[2] One might think that different linear coordinate transformations from τ', ψ' to (τ,ψ) could lead to inequivalent results, but it is easy to show that (19) is the unique possibility, up to trivial scalings and shiftings.

Appropriate choices of rational values for α and γ lead to a countable infinity of solutions for ℓ_2, which in general is real but not necessarily rational, satisfying the condition $0 < \ell_2 < 1/8$ specified in (23).

2.3 Further remarks

The regular Einstein-Sasaki metrics that we have obtained are parameterised by the two rational numbers α and γ, subject only to the condition that ℓ_2 following from (31) satisfy $0 < \ell_2 < 1/8$. In the case where $\ell_1 = \ell_2$, the solutions are included within those discussed in [9].

Although in general ℓ_2 need not be, and indeed is not, rational, special cases can arise where ℓ_2 *is* rational. Since $\ell_1 = \alpha\,\ell_2$, where α must be rational, it follows from (21) that if ℓ_2 is rational then p/q is rational. It also follows from (30) that β, and hence r_+, must then be rational too. Using the scaling symmetry discussed previously, one can then choose p and q to be relatively-prime integers. In the special case with $(p,q) = (1,2)$, the polynomial expression for c^2 in (14) factorises, giving

$$c^2 = -\frac{r(r+\ell_2)(128r^2 + 128r\ell_2 + 64\ell_2^2 - 1)}{(2r+\ell_2)(16r + 8\ell_2 + 1)}\,, \tag{32}$$

and hence

$$r_+ = \frac{\sqrt{2 - 64\ell_2^2} - 8\ell_2}{16}\,. \tag{33}$$

For r_+ to be rational, it is necessary that ℓ, defined by $64\ell^2 + 64\ell_2^2 = 2$, be rational, in which case r_+ is given by

$$r_+ = \tfrac{1}{2}(\ell - \ell_2)\,. \tag{34}$$

Thus the existence of a rational solution amounts to find rational solutions for $64\ell^2 + 64\ell_2^2 = 2$, in which one of ℓ and ℓ_2 must be less than $\frac{1}{8}$, and the other greater than $\frac{1}{8}$. Let ℓ_2 be less than $\frac{1}{8}$. Having a rational solution for $64\ell^2 + 64\ell_2^2 = 2$ is then equivalent to having integer-valued solutions to $x^2 + y^2 = 2z^2$. One can find many integer solutions, by using a computer enumeration, and presumably there are infinitely many. Here, we present a few explicit examples:

$$\begin{aligned}
(\ell_1,\ell_2) &= (\frac{30}{40},\frac{1}{40})\,, &\quad c^2 &= \frac{4r(3-40r)(1+10r)(1+40r)}{5(3+40r)(1+80r)}\\
(\ell_1,\ell_2) &= (\frac{3}{34},\frac{7}{136})\,, &\quad c^2 &= \frac{2r(1-17r)(7+136r)(15+136r)}{17(3+34r)(7+272r)}\,,\\
(\ell_1,\ell_2) &= (\frac{5}{52},\frac{7}{104})\,, &\quad c^2 &= \frac{2r(5-104r)(3+26r)(7+104r)}{13(5+52r)(7+208r)}\,.
\end{aligned} \tag{35}$$

For the cases with $p \neq 2q$, the analysis is much more complicated. For $(p,q) = (1,3)$, we did not find any rational solutions. It is not clear whether such solutions are intrinsically absent, or whether our search was insufficiently exhaustive. For some other values of integer (p,q), we have found isolated rational solutions.

We should again emphasise. however, that the parameters p and q do not need to be rationally related in order that the Einstein-Sasaki metric can be complete.

3 A General Class of Solutions

In this section, we consider a more general class of Einstein-Sasaki metrics in dimension $D = d+1$, constructed as circle bundles over d-dimensional Einstein-Kähler spaces. The d-dimensional Einstein-Kähler space is itself constructed as a complex line bundle over a product of N Einstein-Kähler spaces, with dimensions n_i and metrics $d\Sigma_{n_i}^2$. Thus $d = 2 + \sum_{i=1}^N n_i$, and the d-dimensional Einstein-Kähler metric will be written as

$$ds_d^2 = dt^2 + c^2 \Big(d\tau' - \sum_{i=1}^N p_i A_{(1)}^i\Big)^2 + \sum_{i=1}^N a_i^2 d\Sigma_{n_i}^2\,, \tag{36}$$

where $J_{(2)}^i = dA_{(1)}^i$ is the Kähler form for the Einstein Kähler metric $d\Sigma_{n_i}^2$, with cosmological constant λ_i. The metric (36) is Einstein Kähler with cosmological constant Λ, provided that the functions c and a_i satisfy the first-order equations

$$\frac{\dot a_i}{a_i} = \frac{p_i\, c}{2a_i^2}\,, \qquad \frac{\dot c}{c} = \frac{\lambda_1 - \Lambda a_1^2}{p_1\, c} - \tfrac12 \sum_{i=1}^N \frac{n_i\, \dot a_i}{a_i}\,, \tag{37}$$

together with the set of algebraic constraints

$$\lambda_i\, p_j - \lambda_j\, p_i + \Lambda(\lambda_j\, a_i^2 - \lambda_i\, a_j^2) = 0\,. \tag{38}$$

Note that there are $(N-1)$ independent constraints. The solutions can be obtained straightforwardly, given by

$$a_i^2 = p_i\, r + \ell_i\,, \qquad c^2 = \frac{2}{\prod a_i^{n_i}} \int_0^r \frac{\lambda_1 - \Lambda a_1^2}{p} \prod_i a_i^{n_i}\,, \tag{39}$$

where the coordinate r is defined by $dr = c\,dt$. The integration constants ℓ_i satisfy the constraints

$$\beta = \frac{\Lambda p_i - \lambda_i}{\Lambda\, p_i} = \text{constant}\,, \qquad\qquad \text{for all } i\,. \tag{40}$$

The $D = d+1 = 3 + \sum n_i$ dimensional Einstein-Sasaki metric is given by

$$ds_D^2 = (d\psi' + 2\mathcal{A}_{(1)})^2 + ds_d^2\,, \tag{41}$$

with $\mathcal{A}_{(1)}$ given by

$$\mathcal{A}_{(1)} = r d\tau' - \sum_{i=1}^{N} a_i^2 A^i_{(1)}\,. \tag{42}$$

For the solution to be Einstein, we must have $\Lambda = 4 + \sum n_i$ (after choosing, without loss of generality, $\lambda_i = 1$).

To study the global structure of the metrics, it is appropriate to make the coordinate transformation $\psi' = 2\tau$ and $\tau' = \beta^{-1}(\tau - \Lambda^{-1}\,\psi)$. The metric becomes

$$\begin{aligned} ds_D^2 \;\; = \;\; & \frac{dr^2}{c^2} + \frac{4c^2}{\Lambda^2(c^2+4(\beta+r)^2)}(d\psi - \sum \lambda_i A^i_{(1)})^2 + \sum a_i^2\, d\Sigma^2_{n_i} \\ & + \frac{c^2+4(\beta+r)^2}{\beta^2}\Big(d\tau + \sum \ell_i\, A^i_{(1)} - \frac{c^2+4r(\beta+r)}{\Lambda(c^2+4(\beta+r)^2)}(d\psi - \sum \lambda_i A^i_{(1)})\Big)^2 . \end{aligned} \tag{43}$$

As in the $D = 7$ case we discussed in the previous section, the rate of the collapsing of the circle parameterised by ψ is the same at all the roots of $c^2(r) = 0$. The period of ψ must then be 2π. Consideration of the connection on the fibres parameterised by τ (which never shrink to zero) implies the conditions

$$\begin{aligned} & \frac{\ell_i}{\ell_N} = \alpha_i \equiv \text{rational number}\,, \qquad\qquad i = 1, 2\, \cdots, N-1\,, \\ & \frac{r_+}{\Lambda\,(\beta + r_+)\,\ell_N} = \gamma \equiv \text{rational number}\,. \end{aligned} \tag{44}$$

Note that the p_i do not have to be rationally related, but they satisfy the conditions (40). Substituting (44) and (40) to $c^2 = 0$ equation, we obtain a polynomial equation in ℓ_N of order $1 + \frac{1}{2}\sum n_i$, with rational coefficients that are polynomials in α_i and γ. Without loss of generality, we can choose $0 \le p_1 \le p_2 \le \cdots \le p_N$ and $\lambda_i = 1$. The constant ℓ_N must lie in the range $0 < \ell_N < \Lambda^{-1}$. Thus provided ℓ_N satisfies this condition, the corresponding set of rational numbers (α_i, γ) gives a non-singular Einstein-Sasaki metric.

It is also possible to find special solutions where ℓ_N is rational too. These correspond to cases where the parameters p_i are all relatively-prime integers (after rescaling). Such solutions occur sporadically, and their significance is unclear.

4 Conclusions

In this note, we obtained an infinite number of Einstein-Sasaki metrics in $D = 2n + 3$ dimensions, which are circle bundles over Einstein-Kähler $(2n+2)$-spaces. These spaces are themselves complex line bundles over a product of N Einstein-Kähler manifolds of diverse dimensions n_i. Locally, the Einstein-Sasaki metrics are characterised by N real parameters.

Global considerations for non-singular metrics that extend smoothly onto complete compact manifolds restrict these N parameters to be rational within a certain region.

We focused our attention principally on seven-dimensional examples, which provide natural supersymmetric compactifying manifolds for M-theory.

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Note Added

After this work was completed, a paper appeared that also obtained the local form of the Einstein-Sasaki metrics that we have constructed in this paper [16].

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